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Fluctuations of a Macro-Spin in a Superradiant System

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Fluctuation phenomena in a superradiating atomic system are investigated in the framework of the generalized phase space method. An operator form of the superradiant master equation is mapped onto a *c*-number space. Evolution of the most probable path and small deviations from the path are determined. Fluctuations around the path are solved in a closed form. A remarkable enhancement of fluctuation is observed and this is recognized as a sort of anomalous fluctuation around an unstable point.

KEY WORDS: Superradiance; phase space methods; anomalous fluctuations.

1. INTRODUCTION

The phenomena of superradiance[1] have received renewed attention recently. In the framework of the microscopic theory the so-called superradiant master equation was obtained,[2,3] which describes a temporal evolution of the atomic relaxation process for the density matrix. Thus the problem reduces to a kind of nonlinear spin relaxation. While the ordered distribution function was used to examine the superradiant master equation,[4] recent work[5] has employed a method based on the atomic coherent state.[6] It seems that interesting and important aspects of the problem are fluctuations in the

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superradiant process. Indeed the numerical calculations[7] indicate remarkable fluctuations in the superradiating atomic system. Therefore our main purpose in this work is to treat fluctuation phenomena by an analytical method.

In Section 2, we first develop a systematic approach to the superradiant master equation, applying the phase space method developed earlier,[8,9] which is summarized in the appendix. Making use of the phase space method, we find an equation for the probability distribution function mapped from the operator form of the superradiant master equation. In the same section, evolution of the most probable path and the deviation from the path are determined. Fluctuations around the path are explicitly given. This is performed by paraphrasing the theory recently developed by Kubo *et al.*[10] In Section 3, in order to shed more light on this problem, we discuss various aspects of the behavior of the physical quantities in a moving frame of the most probable path. In the final section, some remarks are presented.

2. SUPERRADIANT MASTER EQUATION AND EVOLUTION OF ATOMIC VARIABLES

In order to investigate the fluctuation phenomena in the superradiant system, we use the phase space method briefly summarized in the appendix.

After elimination of the photon field variables, the atomic system is governed by the following equation for the density matrix[2,3]:

$$\dot{\rho} = \tfrac{1}{2}I_1\{[S_-, \rho S_+] + [S_-\rho, S_+]\} \tag{1}$$

where

$$I_1 = 2g^2/\kappa$$

κ^{-1} is the relaxation time of the photon field and g is the coupling constant between the field and the atomic system.

Let us define a c-number function $F^{(A)}(\theta, \phi)$ from the density matrix ρ based on the anti-normal rule of association. Then Eq. (1) becomes

$$\dot{F}^{(A)}(\theta, \phi) = -\tfrac{1}{2}I_1[(L_x^2 + L_y^2) + \mathbf{L}^2 m_z - i(2S+1)(L_x m_y - L_y m_x)]F^{(A)}(\theta, \phi) \tag{2}$$

It is more appropriate to transform the variables from θ and ϕ to m_x and m_y; the orbital angular momentum operators can be written as

$$L_x = im_z\frac{\partial}{\partial m_y}, \qquad L_y = -im_z\frac{\partial}{\partial m_x}, \qquad L_z = i\left(m_y\frac{\partial}{\partial m_x} - m_x\frac{\partial}{\partial m_y}\right) \tag{3}$$

when acting on $F^{(A)}(\theta, \phi)$. We can easily see that

$$[m_\mu, L_\nu] = im_\lambda \tag{4}$$

where μ, ν, λ, form an even permutation of x, y, z.

Then it is straightforward to rewrite Eq. (2) as

$$\frac{\partial}{\partial\tau}f^{(A)}(m_x, m_y, \tau) = \left\{-\frac{\partial}{\partial m_x} C_{1x} - \frac{\partial}{\partial m_y} C_{1y} + \frac{1}{2}\epsilon\frac{\partial^2}{\partial m_x{}^2} C_{2xx} - \epsilon\frac{\partial^2}{\partial m_x \partial m_y} C_{2xy} + \frac{1}{2}\epsilon\frac{\partial^2}{\partial m_y{}^2} C_{2yy}\right\} f^{(A)}(m_x, m_y, \tau) \quad (5)$$

where we have put

$$\tau = I_1 St, \qquad \epsilon = S^{-1}$$

$$C_{1x} = m_x m_z - \tfrac{1}{2}\epsilon m_x(m_z + 1), \qquad C_{1y} = m_y m_z - \tfrac{1}{2}\epsilon m_y(m_z + 1)$$

$$C_{2xx} = m_z(1 + m_z - m_x{}^2), \qquad C_{2xy} = C_{2yx} = m_x m_y m_z,$$

and

$$C_{2yy} = m_z(1 + m_z - m_y{}^2)$$

In deriving Eq. (5), we have used the relation

$$m_x{}^2 + m_y{}^2 + m_z{}^2 = 1 \quad (6)$$

and the function of $f^{(A)}(m_x, m_y, \tau)$ has been defined by (A.7).

Making use of Eq. (5), we find cumulant equations as

$$\frac{d}{d\tau}\langle m_x\rangle_c = \left(1 - \frac{\epsilon}{2}\right)\langle m_x m_z\rangle_c + \left\{\langle m_z\rangle_c - \frac{\epsilon}{2}(1 + \langle m_z\rangle_c)\right\}\langle m_x\rangle_c \quad (7a)$$

$$\frac{d}{d\tau}\langle m_z\rangle_c = \left(1 - \frac{\epsilon}{2}\right)\langle m_z{}^2\rangle_c - 1 + \langle m_z{}^2\rangle_c - \frac{\epsilon}{2}(1 + \langle m_z\rangle_c)^2 \quad (7b)$$

$$\begin{aligned}\frac{d}{d\tau}\langle m_x{}^2\rangle_c = {}& 2(\langle m_x{}^2 m_z\rangle_c + \langle m_x\rangle_c\langle m_x m_z\rangle_c + \langle m_z\rangle_c\langle m_x{}^2\rangle_c)\\ &-\epsilon\{2\langle m_x{}^2 m_z\rangle_c + (1 + 2\langle m_z\rangle_c)\langle m_x{}^2\rangle_c - \langle m_z{}^2\rangle_c\\ &+ 3\langle m_x\rangle_c\langle m_x m_z\rangle_c - \langle m_z\rangle_c(1 + \langle m_z\rangle_c - \langle m_x\rangle_c{}^2)\}\end{aligned} \quad (7c)$$

$$\begin{aligned}\frac{d}{d\tau}\langle m_z{}^2\rangle_c = {}& 2(\langle m_z{}^3\rangle_c + 2\langle m_z\rangle_c\langle m_z{}^2\rangle_c)\\ &+\epsilon\{-2\langle m_z{}^3\rangle_c - (3 + 5\langle m_z\rangle_c)\langle m_z{}^2\rangle_c\\ &+(1 + \langle m_z\rangle_c)(1 - \langle m_z\rangle_c{}^2)\}\end{aligned} \quad (7d)$$

$$\begin{aligned}\frac{d}{d\tau}\langle m_x m_z\rangle_c = {}& (2\langle m_x m_z{}^2\rangle_c + 3\langle m_z\rangle_c\langle m_x m_z\rangle_c + \langle m_x\rangle_c\langle m_z{}^2\rangle_c)\\ &-\epsilon\{2\langle m_x m_z{}^2\rangle_c + (2 + \tfrac{7}{2}\langle m_z\rangle_c)\langle m_x m_z\rangle_c\\ &+ \tfrac{3}{2}\langle m_x\rangle_c\langle m_z{}^2\rangle_c + \langle m_x\rangle_c\langle m_z\rangle_c(1 + \langle m_z\rangle_c)\}\end{aligned} \quad (7e)$$

$$\frac{d}{d\tau}\langle m_x m_y\rangle_c = (2\langle m_x m_y m_z\rangle_c + \langle m_x\rangle_c\langle m_y m_z\rangle_c + \langle m_y\rangle_c\langle m_z m_x\rangle_c + \langle m_z\rangle_c\langle m_x m_y\rangle_c) - \epsilon\{2\langle m_x m_y m_z\rangle_c + (1 + 2\langle m_z\rangle_c)\langle m_x m_y\rangle_c - \tfrac{3}{2}\langle m_x\rangle_c\langle m_y m_z\rangle_c - \tfrac{3}{2}\langle m_y\rangle_c\langle m_z m_x\rangle_c + \langle m_x\rangle_c\langle m_y\rangle_c\langle m_z\rangle_c\} \tag{7f}$$

and so on. Equations for the m_y are derived by replacing m_x with m_y.

Due to the fact that the characteristic time of a superradiant phenomenon is very short, i.e., $\tau = I_1 St \sim O(1)$, we can prove self-consistently that Eqs. (7a)–(7f) have a solution of the form

$$\langle m_x{}^l m_y{}^m m_z{}^n\rangle_c = \epsilon^{l+m+n-1}A_0 + \epsilon^{l+m+n}A_1 + \cdots \tag{8}$$

Then, putting

$$\langle m_\mu\rangle_c = \overline{m}_\mu + \epsilon u_\mu + O(\epsilon^2), \qquad \langle m_\mu m_\nu\rangle_c = \epsilon\sigma_{\mu\nu} + O(\epsilon^2)$$

we find a set of evolution equations as

$$(d/d\tau)\overline{m}_x = \overline{m}_x\overline{m}_z \tag{9a}$$

$$(d/d\tau)\overline{m}_z = \overline{m}_z{}^2 - 1 \tag{9b}$$

$$(d/d\tau)\, u_x = \overline{m}_z u_x + \overline{m}_x u_z + \sigma_{xz} - \tfrac{1}{2}\overline{m}_x(1 + \overline{m}_z) \tag{10a}$$

$$(d/d\tau)u_z = 2\overline{m}_z u_z + \sigma_{zz} - \tfrac{1}{2}(1 + \overline{m}_z)^2 \tag{10b}$$

$$(d/d\tau)\sigma_{xx} = 2\overline{m}_z\sigma_{xx} + 2\overline{m}_x\sigma_{xz} + \overline{m}_z(1 + \overline{m}_z - \overline{m}_x{}^2) \tag{11a}$$

$$(d/d\tau)\,\sigma_{zz} = 4\overline{m}_z\sigma_{zz} + (1 + \overline{m}_z)(1 - \overline{m}_z{}^2) \tag{11b}$$

$$(d/d\tau)\sigma_{xz} = 3\overline{m}_z\sigma_{xz} + \overline{m}_x\sigma_{zz} - \overline{m}_x\overline{m}_z(1 + \overline{m}_z) \tag{11c}$$

$$(d/d\tau)\sigma_{xy} = 2\overline{m}_z\sigma_{xy} + \overline{m}_x\sigma_{yz} + \overline{m}_y\sigma_{xz} - \overline{m}_x\overline{m}_y\overline{m}_z \tag{11d}$$

Similar equations containing m_y are also obtained. The most probable path of $\langle m_\mu\rangle$ is determined by (9), whereas Eqs. (10) and (11) describe the deviations from the path and the fluctuations around the path, respectively.

Here we must take into account the constraints imposed by (6):

$$\langle m_x{}^2\rangle + \langle m_y{}^2\rangle + \langle m_z{}^2\rangle = 1$$

and

$$\langle m_x{}^2 m_\mu\rangle + \langle m_y{}^2 m_\mu\rangle + \langle m_z{}^2 m_\mu\rangle = \langle m_\mu\rangle, \qquad \mu = x, y, \text{and } z$$

which yield

$$\sum_\mu \overline{m}_\mu{}^2 = 1 \tag{12}$$

$$\sum_\mu (2\overline{m}_\mu u_\mu + \sigma_{\mu\mu}) = 0 \tag{13}$$

and

$$\sum_\mu \overline{m}_\mu\sigma_{\mu\nu} = 0, \qquad \nu = x, y, \text{and } z \tag{14}$$

Let us proceed to solve Eqs. (9)–(11) under the conditions (12)–(14). The most probable path is obtained as

$$\bar{\mathbf{m}} = (\bar{m}_x, \bar{m}_y, \bar{m}_z) = (\sin\theta(\tau)\cos\phi_0, \sin\theta(\tau)\sin\phi_0, \cos\theta(\tau)) \quad (15)$$

where

$$\cos\theta(\tau) = -\tanh\zeta, \qquad \sin\theta(\tau) = \operatorname{sech}\zeta, \qquad \zeta = \tau - \tau_m \quad (16)$$

where τ_m and ϕ_0 represent the initial condition for the polar and azimuthal angles, respectively [$\cos\theta_0 = \tanh\tau_m$, $\theta_0 = \theta(0)$].

Taking into account the symmetry of the system around the z axis, we assume the solution of Eqs. (10) and (11) in the form

$$\begin{aligned}\mathbf{u} &= (u_x, u_y, u_z)\\ &= (v(\zeta)\sin\theta(\tau)\cos\phi_0, v(\zeta)\sin\theta(\tau)\sin\phi_0, v_z(\zeta)\sin^2\theta(\tau))\end{aligned} \quad (17)$$

$$\sigma_{xx} = \sin^2\theta(\tau)\,[\sigma_1(\zeta) + \sigma_2(\zeta)\cos^2\phi_0] \quad (18a)$$

$$\sigma_{yy} = \sin^2\theta(\tau)\,[\sigma_1(\zeta) + \sigma_2(\zeta)\sin^2\phi_0] \quad (18b)$$

$$\sigma_{zz} = \sigma_3(\zeta)\sin^4\theta(\tau) \quad (18c)$$

$$\sigma_{xz} = \sigma_4(\zeta)\sin^3\theta(\tau)\cos\phi_0 \quad (18d)$$

$$\sigma_{yz} = \sigma_4(\zeta)\sin^3\theta(\tau)\sin\phi_0 \quad (18e)$$

and

$$\sigma_{xy} = \sigma_5(\zeta)\sin^2\theta(\tau)\sin\phi_0\cos\phi_0 \quad (18f)$$

The deviations v and v_z and the variances σ_k ($k = 1, 2, \ldots, 5$) are related to one another through

$$\begin{aligned}\sigma_1 + \sigma_2 = \sigma_4\tanh\zeta, &\qquad \sigma_2 = \sigma_5\\ \sigma_4 = \sigma_3\tanh\zeta&\end{aligned} \quad (19)$$

and

$$2\sigma_1 + \sigma_2 + \sigma_3\operatorname{sech}^2\zeta + 2v - 2v_z\tanh\zeta = 0$$

which are direct consequences of (13) and (14). In fact, (19) are consistent with (9)–(11). Therefore we have only three independent equations:

$$(d/d\zeta)(\sigma_1 + \sigma_3) = e^{-2\zeta} \quad (20)$$

$$(d/d\zeta)(\sigma_1 - \sigma_3) = -1 \quad (21)$$

and

$$(d/d\zeta)(2v_z - 2\sigma_3\tanh\zeta + \sigma_3 - \sigma_1 = 0 \quad (22)$$

In order to obtain solutions of (10) and (11), it is sufficient only to solve σ_1, σ_3, and v_z. Thus we have

$$\sigma_1 = \sigma_1(0) - \tfrac{1}{2}(\zeta + \tau_m + \tfrac{1}{2}e^{-2\zeta} - \tfrac{1}{2}e^{2\tau_m}) \tag{23}$$

$$\sigma_3 = \sigma_3(0) + \tfrac{1}{2}(\zeta + \tau_m - \tfrac{1}{2}e^{-2\zeta} + \tfrac{1}{2}e^{2\tau_m}) \tag{24}$$

and from (22), v_z is obtained as

$$v_z = \tfrac{1}{2}\sigma_1 - (\tfrac{1}{2} + \tanh\zeta)\sigma_3 + K \tag{25}$$

K is a constant which is determined later. In this way we have solved the problem completely.

The variances $\sigma_{\mu\nu}$, however, should not be confused with the *real* fluctuations $\zeta_{\mu\nu}$ defined by

$$\epsilon\zeta_{\mu\nu} = \langle(S_\mu - \langle S_\mu\rangle)(S_\nu - \langle S_\nu\rangle) + (S_\nu - \langle S_\nu\rangle)(S_\mu - \langle S_\mu\rangle)\rangle/2S^2$$

Explicitly, these are given by

$$\zeta_{xx} = \sigma_{xx} + \tfrac{1}{2}(1 - \bar{m}_x{}^2) \tag{26}$$

$$\zeta_{xz} = \sigma_{xz} - \tfrac{1}{2}\bar{m}_x\bar{m}_z \tag{27}$$

and

$$\zeta_{zz} = \sigma_{zz} + \tfrac{1}{2}(1 - \bar{m}_z{}^2) \tag{28}$$

In deriving these expressions, we have used phase space forms of the spin operators, for example,[9]

$$S_x \to Sm_x \tag{29}$$

$$S_x{}^2 \to S(S - \tfrac{1}{2})m_x{}^2 + \tfrac{1}{2}S \tag{30}$$

Thus we have obtained ζ's in an analytical form. Figure 2 shows some examples of the fluctuations. More details will be given in the next section.

3. FLUCTUATIONS IN A MOVING FRAME

In order to clarify the results obtained in the preceding section, we shall discuss the variances and the deviations in the system moving along with the classical spin $(\bar{m}_x(\tau), \bar{m}_y(\tau), \bar{m}_z(\tau))$. Then $(m_{x'}, m_{y'}, m_{z'})$ in the new coordinate system (see Fig. 1) is represented by

$$\begin{pmatrix} m_{x'} \\ m_{y'} \\ m_{z'} \end{pmatrix} = \begin{bmatrix} \sin\theta(\tau)\cos\phi_0 & \sin\theta(\tau)\sin\phi_0 & \cos\theta(\tau) \\ -\sin\phi_0 & \cos\phi_0 & 0 \\ -\cos\theta(\tau)\cos\phi_0 & -\cos\theta(\tau)\sin\phi_0 & \sin\theta(\tau) \end{bmatrix} \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} \tag{31a}$$

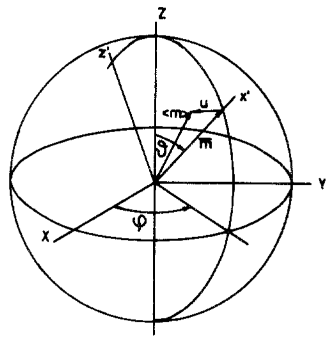

Fig. 1. Representation of the most probable value $\bar{\mathbf{m}}$ and the deviation $\mathbf{u}$ in polar coordinates.

the inverse transformation of which is given by

$$\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \begin{bmatrix} \sin\theta(\tau)\cos\phi_0 & -\sin\phi_0 & -\cos\theta(\tau)\cos\phi_0 \\ \sin\theta(\tau)\sin\phi_0 & \cos\phi_0 & -\cos\theta(\tau)\sin\phi_0 \\ \cos\theta(\tau) & 0 & \sin\theta(\tau) \end{bmatrix} \begin{pmatrix} m_{x'} \\ m_{y'} \\ m_{z'} \end{pmatrix} \quad (31b)$$

Using (31a), the variances and the deviations in this frame can be calculated, for example, as

$$\langle m_{x'} m_{x'} \rangle_c = \sum_{\mu,\nu} \bar{m}_\mu \bar{m}_\nu \langle m_\mu m_\nu \rangle_c = \epsilon \sum_{\mu,\nu} \bar{m}_\mu \bar{m}_\nu \sigma_{\mu\nu} + O(\epsilon^2)$$

Hence we have

$$\sigma_{x'x'} = \sum_{\mu,\nu} \bar{m}_\mu \bar{m}_\nu \sigma_{\mu\nu} = 0$$

where we have used (14). In the same way, we obtain

$$\sigma_{x'z'} = \sigma_{y'z'} = \sigma_{x'y'} = 0 \quad (32)$$

$$\sigma_{x'x'} = 0 \quad (33a)$$

$$\sigma_{y'y'} = \sigma_1 \sin^2\theta(\tau) \quad (33b)$$

$$\sigma_{z'z'} = \sigma_3 \sin^2\theta(\tau) \quad (33c)$$

$$\bar{m}_{x'} = 1, \qquad \bar{m}_{y'} = \bar{m}_{z'} = 0 \quad (34)$$

$$u_{x'} = -\tfrac{1}{2}(\sigma_{y'y'} + \sigma_{z'z'}) \quad (35a)$$

$$u_{y'} = 0 \quad (35b)$$

and

$$u_{z'} = \sin\theta(\tau)\,\{(\sigma_1 - \sigma_3)\cos^2\theta(\tau)/2 - [\sigma_1(0) - \sigma_3(0)]\cos^2\theta(0)/2\} \quad (35c)$$

Now the meaning implied by the conditions (6) is clear. That is, it is equivalent to the following statements:

(i) In the frame moving along with the classical spin, there are no cross correlations or fluctuations in the radial direction.

(ii) The effective length of the spin and the variances are related to each other through (35a); as the fluctuations become larger, the average length of the spin becomes shorter and vice versa.

The initial conditions are determined by requiring that the polar and the azimuthal angles of $\langle m \rangle$ coincide with those of the classical spin $\bar{m}$, i.e., $u_{z'} = 0$ at $\tau = 0$ (see Fig. 1). Then the constant K in (25) is determined to be

$$K = \tfrac{1}{2}(1 + \tanh \tau_m)[\sigma_1(0) - \sigma_3(0)]$$

and hence we can freely choose the initial conditions on ϕ_0, $\theta(0)$, $\sigma_{y'y'}(0)$, and $\sigma_{z'z'}(0)$. Initial values of other quantities are related through (19).

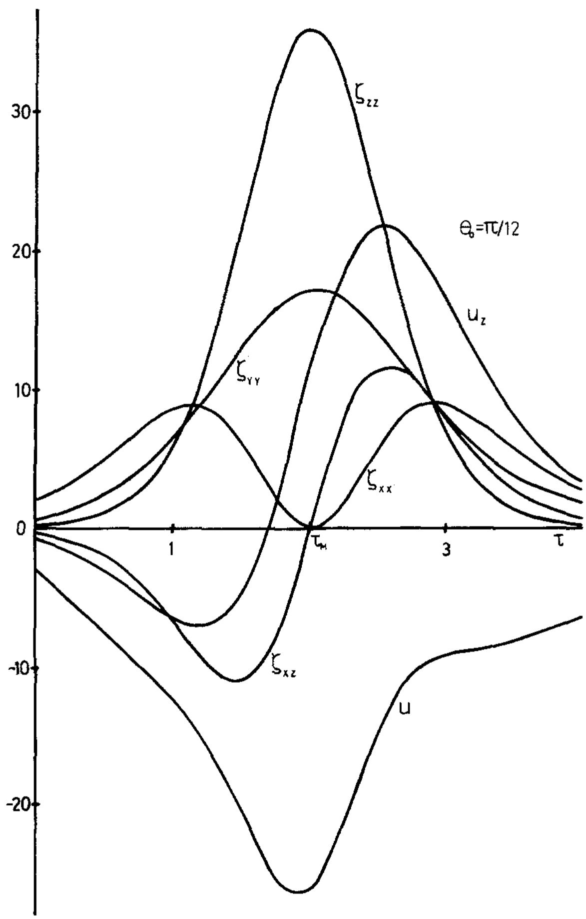

Fig. 2. Typical behavior of the various quantities as a function of τ for $\theta_0 = \pi/12$.

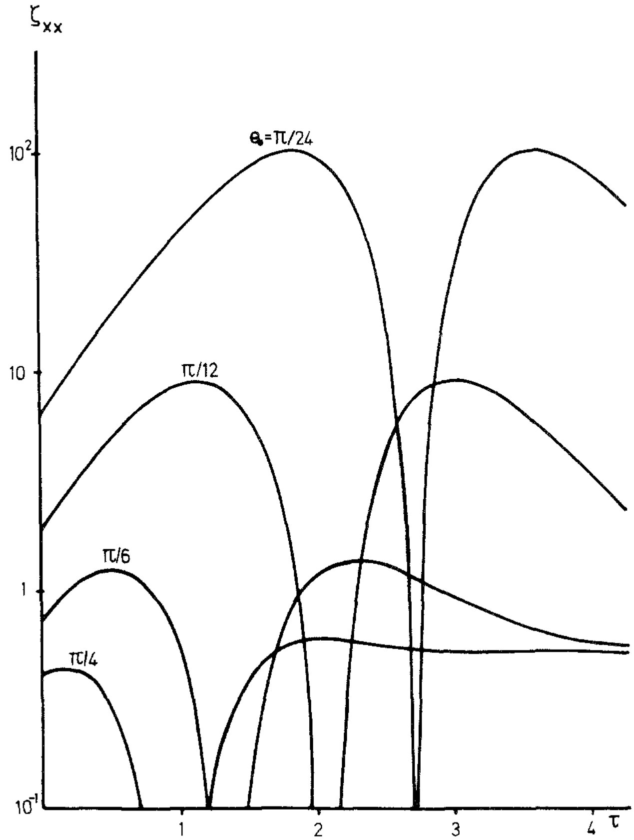

Fig. 3. The transverse fluctuation ζ_{xx} as a function of τ for various initial values of θ_0.

The variances and the deviations in the old frame are expressed in terms of those in the moving frame as

$$\begin{aligned}
\sigma_{xx} &= \sigma_{y'y'} \sin^2 \phi_0 + \sigma_{z'z'} \cos^2 \theta(\tau) \cos^2 \phi_0 \\
\sigma_{yy} &= \sigma_{y'y'} \cos^2 \phi_0 + \sigma_{z'z'} \cos^2 \theta(\tau) \sin^2 \phi_0 \\
\sigma_{zz} &= \sigma_{z'z'} \sin^2 \theta(\tau) \\
\sigma_{xz} &= -\sigma_{z'z'} \sin \theta(\tau) \cos \theta(\tau) \cos \phi_0 \\
\sigma_{yz} &= -\sigma_{z'z'} \sin \theta(\tau) \cos \theta(\tau) \sin \phi_0 \\
\sigma_{xy} &= -\sin \phi_0 \cos \phi_0 \, [\sigma_{y'y'} - \sigma_{z'z'} \cos^2 \theta(\tau)] \\
u_x &= \cos \phi_0 [u_{x'} \sin \theta(\tau) - u_{z'} \cos \theta(\tau)] \\
u_y &= \sin \phi_0 \, [u_{x'} \sin \theta(\tau) - u_{z'} \cos \theta(\tau)] \\
u_z &= u_{x'} \cos \theta(\tau) - u_{z'} \sin \theta(\tau)
\end{aligned} \tag{36}$$

In this way, we are naturally led to the expressions (18) again. We can see that the variances σ_1 and σ_2 defined there are directly related to the fluctuations in the moving frame through (33b) and (33c).

We plot ζ_{xx}, ζ_{yy}, ζ_{zz}, and ζ_{xz} as functions of τ in Fig. 2. The deviations u_z and u_x are also plotted in Fig. 2 as functions of τ. The fluctuations ζ_{xx} and ζ_{zz} are plotted for various initial conditions θ_0 in Figs. 3 and 4, respectively. The variances $\zeta_{z'z'}$ and $\zeta_{y'y'}$ and the deviation $u_{x'}$ in the moving frame are shown in Fig. 5 as functions of $\theta(\tau)$. We find some interesting features from these figures. For example, ζ_{xx} vanishes at $\theta = \pi/2$, while ζ_{xz} changes its sign there. This behavior is attributed to the factors appearing in (36) and also the consequences deduced from the earlier statement (i). The deviation $u_{x'}$ becomes large near $\theta(\tau) \sim \pi/2$ due to the large fluctuations of the ζ's.

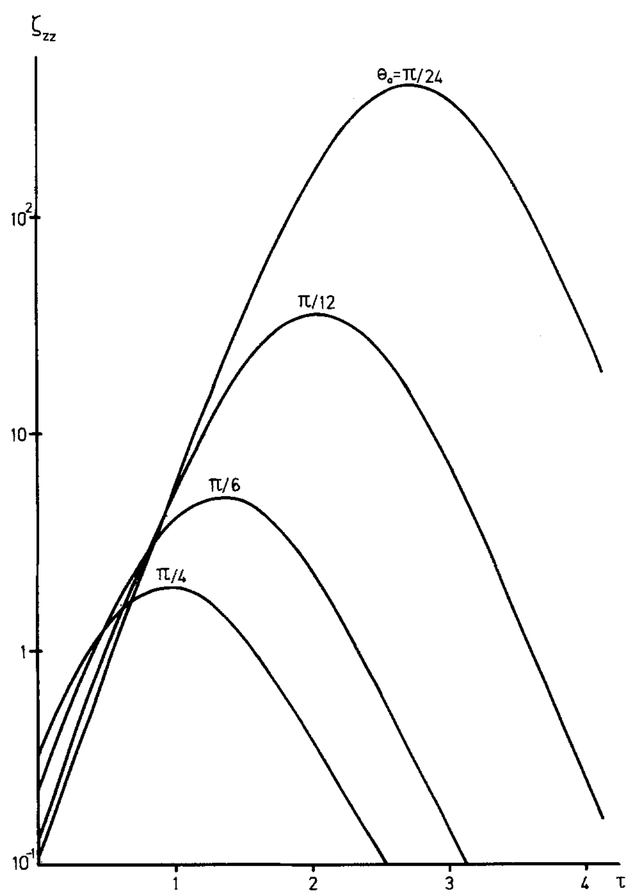

Fig. 4. The longitudinal fluctuation ζ_{zz} as a function of τ. The initial values of θ_0 are the same as those for ζ_{xx}.

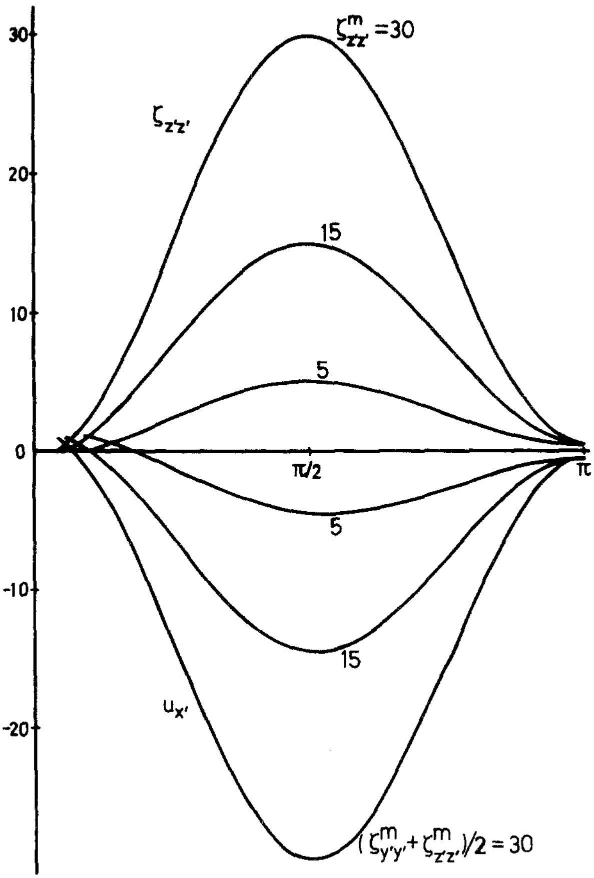

Fig. 5. The longitudinal fluctuation $\zeta_{z'z'}$ and the transverse deviation $u_{x'}$ in the moving frame as a function of $\theta(\tau)$.

As $\theta(\tau) \to \pi$ $(\tau \to \infty)$, $\zeta_{z'z'}$ and $\zeta_{y'y'}$ tend to a constant $\frac{1}{2}$, corresponding to the fact that the spin relaxes to the angular momentum eigenstate $|S, -S)$.

Before concluding this section, we discuss the dependence of the maximum value and the half-width of variances on the initial conditions. When we release the spin from the upper half-sphere, the variances $\zeta_{z'z'}$ and $\zeta_{y'y'}$ have their maximum near $\theta(\tau) = \pi/2$. These variances are given by

$$\begin{aligned} \zeta_{z'z'} &= \tfrac{1}{2}[\sin^2\theta(\tau)\ln\tan\tfrac{1}{2}\theta(\tau) - \cos\theta(\tau)] + \zeta^m_{z'z'}\sin^2\theta(\tau) \\ \zeta_{y'y'} &= -\tfrac{1}{2}[\sin^2\theta(\tau)\ln\tan\tfrac{1}{2}\theta(\tau) + \cos\theta(\tau)] + \zeta^m_{y'y'}\sin^2\theta(\tau) \end{aligned} \tag{37}$$

where we have defined $\zeta^m_{z'z'}$ and $\zeta^m_{y'y'}$ by

$$\begin{aligned} \zeta^m_{z'z'} &= \sigma_3(0) + \tfrac{1}{4}(\ln\cot^2\tfrac{1}{2}\theta_0 + \cot^2\tfrac{1}{2}\theta_0 + 1) \\ \zeta^m_{y'y'} &= \sigma_1(0) + \tfrac{1}{4}(-\ln\cot^2\tfrac{1}{2}\theta_0 + \cot^2\tfrac{1}{2}\theta_0 + 1) \end{aligned} \tag{38}$$

We plot $\zeta^m_{z'z'}$ and $\zeta^m_{y'y'}$ as functions of θ_0 in Fig. 6; $\zeta^m_{z'z'}$ and $\zeta^m_{y'y'}$, in general, will not give the correct maximum values, but as $\theta_0 \to 0$, the first terms in (37) can be neglected, and therefore the maximum values are approximated by them.

A half-width Δ can be defined as the difference of the angles θ_1 and θ_2 where ζ attains the value $\zeta^m/2$. Changing θ's into α's through the relation $\theta = \pi/2 - \alpha$, we have α_1 and α_2 as the solutions of the following equations:

$$\zeta^m_{z'z'} = \frac{1}{2(1 - 2\cos^2\alpha)}\left[\cos^2\alpha \ln\frac{1 - \sin\alpha}{1 + \sin\alpha} - 2\sin\alpha\right]$$

and

$$\zeta^m_{y'y'} = \frac{1}{2(2\cos^2\alpha - 1)}\left[\cos^2\alpha \ln\frac{1 - \sin\alpha}{1 + \sin\alpha} + 2\sin\alpha\right]$$

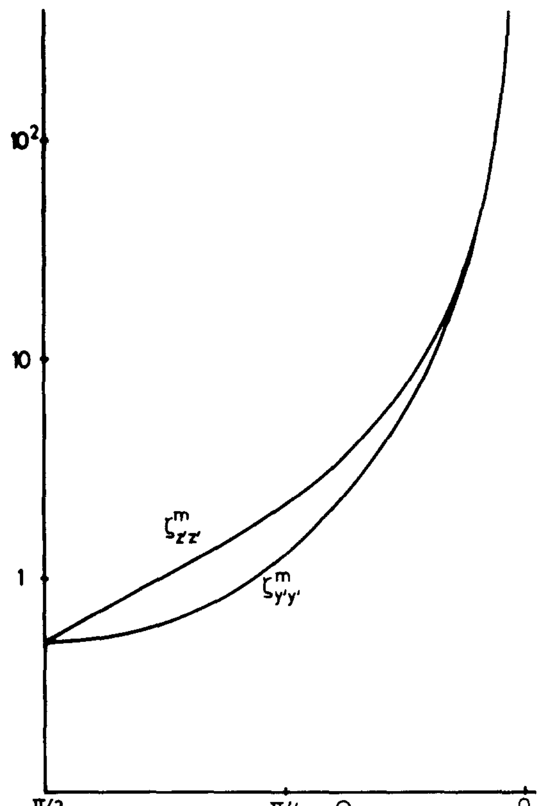

Fig. 6. Measure of the maximum values of the fluctuations $\zeta_{y'y'}$ and $\zeta_{z'z'}$ as a function of the initial value θ_0.

First we note from (38) that the ζ^m become exceedingly large as the initial direction of the spin approaches the north pole (see Fig. 6). This is a sort of anomalous fluctuation around the unstable point.[10]

As for the width, Δ decreases as ζ^m increases, and tends to $\pi/2$ as $\zeta^m \to \infty$.

4. CONCLUDING REMARKS

We have investigated fluctuation phenomena of a macro-spin under the influences of a radiation field. This system gives an interesting example of a stochastic motion of spins. We found remarkable fluctuations when the spin passes through the superradiant state ($\theta = \pi/2$) and interesting behavior of the transverse fluctuations.

Our findings afford a solvable, typical system which shows anomalous fluctuations around the unstable point.[10]

APPENDIX

In this appendix, a brief summary is given of the generalized phase space method of the spin systems.[8,9]

An operator G is mapped onto a c-number function $F^{(N)}(\theta, \phi)$ (normal rule of association) defined as a diagonal element of the Bloch state $|S; \omega\rangle$:

$$F^{(N)}(\theta, \phi) = \langle S; \omega|G|S; \omega\rangle \tag{A.1}$$

or onto a function $F^{(A)}(\theta, \phi)$ (anti-normal rule of association) obtained by expressing the operator in the form

$$G = (2S + 1) \int \frac{d\omega}{4\pi} |S; \omega\rangle\langle S; \omega| F^{(A)}(\theta, \phi) \tag{A.2}$$

where ω denotes a solid angle spanned by θ and ϕ.

The trace of two operators G_1 and G_2 is expressed by

$$\mathrm{Tr}\, G_1 G_2 = (2S + 1) \int \frac{d\omega}{4\pi} F_1^{(N)}(\theta, \phi) F_2^{(A)}(\theta, \phi) \tag{A.3}$$

The following theorem turns out to be very useful for obtaining a c-number form of an equation for the density matrix ρ.

Products of operators $S_\mu G$ and GS_μ are mapped onto a c-number space according to the rule given by

$$\begin{aligned} S_\mu G &\to \mathscr{S}_\mu^{(\Omega)} F^{(\Omega)}(\theta, \phi) \\ GS_\mu &\to \mathscr{S}_\mu^{(\Omega)*} F^{(\Omega)}(\theta, \phi), \qquad \mu = x, y, \text{ and } z \end{aligned} \tag{A.4}$$

where the superscript Ω specifies a mapping rule. Explicitly they are given by

$$\mathscr{S}^{(\mathrm{N})} = S\mathbf{m} + \tfrac{1}{2}\mathbf{L} - \tfrac{1}{2}i(\mathbf{m} \times \mathbf{L}) \tag{A.5a}$$

for the normal rule of association, and

$$\mathscr{S}^{(\mathrm{A})} = (S + 1)\mathbf{m} + \tfrac{1}{2}\mathbf{L} + \tfrac{1}{2}i(\mathbf{m} \times \mathbf{L}) \tag{A.5b}$$

for the anti-normal rule of association, respectively. The pseudo-spin $\mathbf{m}$ is defined by

$$\mathbf{m} = (m_x, m_y, m_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

and $\mathbf{L}$ is the "orbital angular momentum" operator defined by (3).

From (A.3), a statistical average is obtained in this phase space as

$$\langle O(m_x, m_y, t)\rangle = \int dm_x\, dm_y\, f^{(\mathrm{A})}(m_x, m_y, t) O(m_x, m_y) \Big/ \int dm_x\, dm_y\, f^{(\mathrm{A})}(m_x, m_y, t) \tag{A.6}$$

where the quasi-probability distribution function is introduced by

$$f^{(\mathrm{A})}(m_x, m_y, t) = F^{(\mathrm{A})}(m_x, m_y, t)/m_z \tag{A.7}$$

the function $F^{(\mathrm{A})}(\theta, \phi)$ being obtained by mapping the density matrix.

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